AIAA JOURNAL
Vol. 31, No. 9, September 1993

Comparison of Advanced Reduced-Basis Methods for Transient Structural Analysis

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Two advanced reduced-basis methods, the force-derivative method and the Lanczos method, are compared with two widely used modal methods, the mode-displacement method and the mode-acceleration method. Three example structural problems are studied: an undamped and a discretely damped multispan beam subject to a uniformly distributed load that varies as a quintic function of time, and a high-speed civil transport subject to a normal wing-tip load that varies as a sinusoidal function of time. The comparison criteria include the number of basis vectors required to obtain a desired level of accuracy and the associated computational times. It is shown that, in general, the force-derivative method obtains converged solutions using the fewest number of basis vectors and the smallest amount of CPU time. However, for structures where only a small number of modes participate in the response, all of the methods require very few basis vectors for convergence, and the relative advantages of the higher order modal methods are not as significant. Although the Lanczos method was shown to be effective in solving the two undamped example problems, it was not effective in solving the discretely damped multispan beam example.

Nomenclature

C = damping matrix, $n \times n$
E = modulus of elasticity
e = spatial error norm, see Eq. (11)
f = force vector, $n \times 1$
I = identity matrix, moment of inertia
K = stiffness matrix, $n \times n$
L = length
M = mass matrix, $n \times n$, moment
m = subset of total number of degrees of freedom
n = total number of degrees of freedom
Q = matrix of Lanczos vectors
q_i = ith Lanczos vector
T_m = tridiagonal matrix generated by Lanczos-vector-based eigensolver
t = time
u = displacement vector, $n \times 1$
$\dot{u}$ = velocity vector, $n \times 1$
$\ddot{u}$ = acceleration vector, $n \times 1$
x = vector of generalized coordinates
x_i = ith generalized coordinate
Y = matrix of basis vectors
y_i = ith basis vector
α, β = Newmark-Beta method integration constants
Δt = time step used in Newmark-Beta integration
δ = $u_f - u_a$
ζ_i = ith modal viscous damping factor
Λ = matrix of damping coefficients, $\Phi^T C \Phi$, see Eq. (8)
ρ = density
Φ = matrix of undamped eigenvectors
ϕ_i = ith undamped eigenvector
Ω = matrix of undamped natural frequencies
ω_i = ith undamped natural frequency

Subscripts

a = approximate solution, see Eq. (11)
f = full-system solution, see Eq. (11)
S = stress

Superscripts

T = transpose
$\dot{}$ = derivative with respect to time
$\hat{}$ = subset of the matrix of basis vectors, eigenvalues, or generalized coordinates

Presented as Paper 91-1059 at the AIAA/ASME/ASCE/AHS/ASC 32nd Structures, Structural Dynamics, and Materials Conference, Baltimore, MD, April 8–10, 1991; received Sept. 24, 1991; revision received March 10, 1993; accepted for publication March 12, 1993.

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Introduction

THE transient analysis of complex structures that are modeled as discrete, multi-degree-of-freedom systems often requires the solution of very large, coupled systems of equations. Calculation of the transient structural response for such large systems is computationally expensive, and hence methods that can significantly reduce the problem size and computational cost and still retain solution accuracy are highly desirable. A class of methods, called reduced-basis methods, has been developed that attempts to approximate the solution of the complete system of equations using a much smaller or reduced set of generalized basis vectors.[1-5] Examples of basis vectors that have been investigated include eigenvectors,[1-4] Ritz vectors,[5] Lanczos vectors,[6] and combinations of the aforementioned.[7] A reduced-basis method that uses the eigenvectors of a system is referred to herein as a modal method. Similarly, a reduced-basis method that uses Lanczos vectors as basis vectors will be referred to as the Lanczos method.

Two of the most widely used modal methods for transient structural analysis are the mode-displacement method (MDM) and the mode-acceleration method (MAM). The MDM has been shown to be effective in accurately predicting the displacement response of many complex structures using relatively few eigenvectors; however, the MDM has not been as effective in accurately predicting the stress solution. The MAM was developed[3,4] as a means for improving the convergence of the stress solutions from the MDM. An in-depth comparison of the MDM and the MAM[4] indicates that the MAM converges to an accurate solution using fewer modes than the MDM.

The use of Ritz and Lanczos vectors as basis vectors was investigated because of their good convergence characteristics compared with the MDM and because of the relative computational ease in computing the basis vectors compared with computing the eigenvectors. Recent advances in computational methods and algorithms for computing the Lanczos vectors[8,9] have increased interest in this method for reducing the computational cost of solving large, transient structural problems.

Recently, a higher order modal method, entitled the force-derivative method (FDM), has been developed[10,11] to increase the convergence rate of the MDM and the MAM. It was found in Ref. 11 that the FDM is a means for unifying all modal methods and, in particular, for developing any desired order of a higher order modal method. Using the FDM formulation, the MDM can be classified as a zeroth-order method and the MAM as a first-order method. It was also found in Ref. 11 that the FDM consistently resulted in more accurate solutions than either the MDM or the MAM and that the FDM required fewer modes or degrees of freedom to converge to an acceptable solution.

Most studies to date compare the number of basis vectors required for convergence of several reduced-basis methods. It is rare that comparisons are made of computational time vs accuracy for the various methods. A study was recently conducted[12] that compared the FDM, the Lanczos method, the MDM, and the MAM using both of these criteria. The purpose of the present study is to extend the work presented in Ref. 12 by analyzing a larger, more realistic example problem, a high-speed civil transport structure. The number of basis vectors required to achieve a desired level of accuracy, as well as the associated computational time of each method, is presented in this paper. The approximate solutions obtained using the reduced-basis methods are compared with a full-system solution calculated by integrating directly the full system of equations of motion using the Newmark-Beta implicit time-integration scheme. The four reduced-basis methods (MDM, MAM, FDM, and Lanczos method) have been implemented on a CONVEX C220 high-performance computer and incorporated into the computational mechanics testbed (COMET)[13] general-purpose finite element code.

Determination of the Full-System Solution

The approximate solutions obtained using the reduced-basis methods are compared with a full-system solution obtained by integrating the full system of equations of motion directly using the Newmark-Beta implicit time-integration scheme. The equations of motion to be solved represent a damped, linear, structural system with n degrees of freedom and can be written in discrete form as

$$M\ddot{u} + C\dot{u} + Ku = f(t) \tag{1}$$

This system of equations is solved at discrete times using the Newmark-Beta method as described in Refs. 14 and 15 to yield the full-system solution. The values of the integration constants α and β are assumed in the present method to be 0.25 and 0.50, respectively. Therefore, the method is equivalent to the trapezoidal rule or the constant-average-acceleration method.[14,15] The time step Δt is equal to 0.0001. This value was selected by carrying out several analyses of a cantilevered beam example problem while successively decreasing the time step until no appreciable change in the full-system solution was apparent. For large problems, the direct integration of the full system of equations can require a significant amount of computational effort.

Review of the General Theory of Reduced-Basis Methods

In using a reduced-basis method, the desired response (displacements in this case) can be represented by the superposition of a set of linearly independent basis vectors scaled by a set of time-dependent generalized coordinates, as shown in Eq. (2):

$$u(t) = \sum_{i=1}^{n} y_i x_i(t) = Yx(t) \tag{2}$$

The key to reduced-basis method theory is that the solution of the full system of equations [Eq. (2)] can be adequately approximated using a significantly reduced set of m basis vectors, as shown in Eq. (3):

$$u(t) \approx \sum_{i=1}^{m} y_i x_i(t) = \hat{Y}\hat{x}(t) \qquad m \ll n \tag{3}$$

By carrying out the appropriate transformation for the reduced-basis method being used, the following reduced system of $m \times m$ equations is obtained,

$$\bar{M}\ddot{\hat{x}}(t) + \bar{C}\dot{\hat{x}}(t) + \bar{K}\hat{x}(t) = \bar{f}(t) \qquad m \times m \tag{4}$$

where the barred terms depend on the specific method used and will be defined subsequently. The generalized coordinates $\hat{x}(t)$ may then be obtained by solving Eq. (4) in a step-by-step manner with a time-integration scheme such as the Newmark-Beta method. Results for $\hat{x}(t)$ are then substituted into Eq. (3) to approximate the response. Through a judicious choice of basis vectors and solution algorithms, the problem size and computational time can be greatly reduced compared with the solution of the full system of equations of motion.

Modal Methods

For the MDM, the displacements $u(t)$ are approximated by the superposition of the eigenvectors of the system scaled by a set of generalized coordinates known as modal coordinates, as shown in Eq. (5):

$$u(t) \approx \sum_{i=1}^{m} \phi_i x_i(t) = \hat{\Phi}\hat{x}(t) \qquad \text{order} \quad 0 \tag{5}$$

For structural systems where M, C, and K are symmetric, the modal coordinates are obtained by solving Eq. (4) with the barred terms defined by Eqs. (6):

$$\bar{M} = \hat{\Phi}^T M \hat{\Phi} = [I] \tag{6a}$$

$$\bar{C} = \hat{\Phi}^T C \hat{\Phi} \tag{6b}$$

$$\bar{K} = \hat{\Phi}^T K \hat{\Phi} \tag{6c}$$

$$\bar{f}(t) = \hat{\Phi}^T f(t) \tag{6d}$$

The displacement response may then be obtained by substituting the modal coordinates into Eq. (5). If the system is undamped or proportionally damped, Eq. (4) results in an uncoupled system of equations, and if the damping is nonproportional, the resulting equations are coupled through the damping matrix.[1]

The MAM was developed to improve the convergence of the stress solutions from the MDM. The MAM is a first-order method that contains the results from the zeroth order MDM plus an additional term and, as shown in Ref. 11, can be put into the following form:

$$u(t) = \hat{\Phi}\hat{x}(t) + (K^{-1} - \hat{\Phi}\hat{\Omega}^{-2}\hat{\Omega}^T)f(t) \qquad \text{order} \quad 1 \tag{7}$$

The additional term is sometimes referred to as a pseudostatic correction term[14] because it contains the static displacement of the system $[K^{-1}f(t)]$ and an additional correction factor. This term serves to approximate the flexibility of the higher modes that are neglected in the summation in Eq. (5).

The FDM is a higher order modal method that was developed to improve the convergence rates of the MDM and the MAM. A unified derivation of all of these modal methods that clarifies the mathematical relationship between them is presented in Refs. 10 and 11. As such, the MDM may be considered to be a zeroth order form of the FDM, and the MAM may be considered to be a first-order form of the FDM. A second-order expression of the FDM is given in Eq. (8):

$$u(t) \approx \hat{\Phi}\hat{x}(t) + (K^{-1} - \hat{\Phi}\hat{\Omega}^{-2}\hat{\Phi}^T)f(t)$$
$$- (K^{-1}CK^{-1} - \hat{\Phi}\hat{\Omega}^{-2}\hat{\Lambda}\hat{\Omega}^{-2}\hat{\Phi}^T)\dot{f}(t) \qquad \text{order } 2 \tag{8}$$

As shown in Eq. (8), the second-order form consists of the first-order MAM plus a higher order term multiplied by the first time derivative of the forcing function. As in the case of the MAM, the additional terms serve to approximate better the flexibility of the higher modes that are neglected by the use of a reduced set of modes. Furthermore, increasing the order of the approximation results in the addition of higher order terms multiplied by successively higher order time derivatives of the forcing function.[11] Hence, the order of the FDM used is dependent on the number of time derivatives of the forcing function that exist. For example, if the forcing function is described by a quadratic function of time (for which there exists a maximum of two nonzero time derivatives), the highest-order FDM that will yield useful results is order 3. If an order higher than 3 is used, all of the additional terms (past order 3) would be zero. The form of the FDM used in Eq. (8) was shown in Ref. 11 to be valid for both proportional and nonproportional damping.

Lanczos Method

The Lanczos method uses Lanczos vectors as an alternative to using undamped eigenvectors as basis vectors. The Lanczos vectors are obtained using the Lanczos algorithm as described in Refs. 6, 16, and 17. In addition to the Lanczos vectors, the Lanczos algorithm produces a tridiagonal matrix[8,9] T_m of order m. The use of the Lanczos vectors and T_m in the transient analysis is described subsequently.

For this method, the displacements $u(t)$ are approximated by the superposition of the Lanczos vectors scaled by a set of generalized coordinates known as Lanczos coordinates:

$$u(t) \approx \sum_{i=1}^{m} q_i x_i(t) \approx \hat{Q}\hat{x}(t) \qquad m \ll n \tag{9}$$

Applying the transformation described in Ref. 6, the reduced system of equations given in Eq. (4) may be obtained, with the barred terms defined as

$$\bar{M} = \hat{Q}^T MK^{-1}M\hat{Q} = T_m \tag{10a}$$

$$\bar{C} = \hat{Q}^T MK^{-1}C\hat{Q} \tag{10b}$$

$$\bar{K} = \hat{Q}^T M\hat{Q} = I \tag{10c}$$

$$\bar{f}(t) = \hat{Q}^T MK^{-1}f(t) \tag{10d}$$

As with the modal methods, the Lanczos coordinates may be obtained by solving Eq. (4) with the barred terms as defined by Eqs. (10). The Lanczos coordinates are then multiplied by the Lanczos vectors and summed using Eq. (9) to approximate the displacements. If the system is undamped, Eq. (4) results in a tridiagonal system of equations, and the computational size of the problem is therefore reduced. Furthermore, if the damping is a special case of proportional damping called Rayleigh damping, in which the damping matrix is a linear combination of the mass and stiffness matrices (i.e., $C = a_0M + a_1K$), the tridiagonal form is preserved.[6] Other forms of proportional damping and nonproportional damping do not preserve the tridiagonal form of the equations. The Lanczos method has been considered to be an efficient method for predicting transient structural behavior because the computational effort required to generate and orthogonalize the Lanczos vectors is typically less than that required to compute the natural modes of the system, which are required for use with the modal methods.

Determination of the Basis Vectors

The four reduced-basis methods discussed in this paper, as well as the direct integration of the full system of equations of motion by the Newmark-Beta time-integration scheme, have been implemented on high-performance computers and incorporated into COMET, a general-purpose finite element code.[13] The computational times presented are for a CONVEX C220 high-performance computer.

The Lanczos vectors are computed by an eigensolver that is based on the Lanczos algorithm described in Refs. 8 and 9. Furthermore, it was shown in Refs. 8 and 9 that computing the natural frequencies and eigenvectors of a structural system from a set of Lanczos vectors is much more computationally efficient than using subspace iteration. The Lanczos-vector-based eigensolver described in Refs. 8 and 9 was implemented to take advantage of the capabilities of high-performance computers, particularly vector capabilities; therefore, it was also used to compute the natural frequencies and eigenvectors for the example problems presented in this paper.

Spatial Error Norm

The accuracy of the transient response calculated using the reduced-basis methods was evaluated quantitatively using a relative, spatial error norm defined in Refs. 10 and 11. Since the error norm selected is a spatial error norm, it is calculated at only one point in time. The error norm used to evaluate the approximate displacement solutions (referred to as the displacement error norm) is defined in Eq. (11):

$$e_u = \sqrt{\frac{(\delta^T\delta)}{(u_f^T u_f)}} \tag{11}$$

Similarly, a moment error norm e_M and a stress norm e_s may be obtained using Eq. (11) and replacing the displacement vectors with vectors containing either nodal moment values or values of stress. The nodal moment values and the stress values for all of the methods (including the full-system solution) are computed by back-substituting the displacement response solutions into a stress postprocessor in the COMET code.

Computational Procedures

The first step in the MDM is to calculate the modal coordinates $\hat{x}(t)$ by integrating Eq. (4). The integration method used is identical to the method used to calculate the full-system solution. The displacements $u(t)$ are then computed using Eq. (5).

The first step in the MAM is to factor the stiffness matrix K. Solving the equation $Kr = f(t)$ for r yields the term $K^{-1}f(t)$. The product $\hat{\Phi}\hat{\Omega}^{-2}\hat{\Phi}^T f(t)$ is calculated using matrix-vector multiplication, taking into account that the $\hat{\Omega}^{-2}$ matrix is diagonal. The displacements $u(t)$ are then computed using Eq. (7).

The first step in the calculation of the second-order term for the FDM (order 2) is to solve the equation $Kr = \dot{f}(t)$ to yield $K^{-1}\dot{f}(t)$. Next, the equation $Kr = CK^{-1}\dot{f}(t)$ is solved to yield the term $K^{-1}CK^{-1}\dot{f}(t)$. The term $\hat{\Phi}\hat{\Omega}^{-2}\hat{\Lambda}\hat{\Omega}^{-2}\hat{\Phi}\dot{f}(t)$ is computed by matrix-vector multiplication. If the damping is proportional, computational expense is reduced by taking into account that the $\hat{\Lambda}$ matrix is diagonal. The displacements $u(t)$ are then computed using Eq. (8). Higher order terms for the FDM are computed using a similar procedure.

The computational procedure for the Lanczos method is similar to that described for the FDM (order 2). The first step in the Lanczos method is to factor the stiffness matrix K. The equation $Kr = C\hat{Q}$ is then solved to yield $K^{-1}C\hat{Q}$. Then, $\bar{C} = \hat{Q}^T MK^{-1}C\hat{Q}$ can be calculated using matrix-matrix multiplication. Similarly, $\bar{f}(t)$ may be calculated by solving the equation $Kr = f(t)$ to yield $K^{-1}f(t)$ and then premultiplying by $\hat{Q}^T M$ to yield $\bar{f}(t) = \hat{Q}^T M K^{-1} f(t)$. Once these terms are calculated, the Lanczos coordinates $\hat{x}(t)$ are then computed by integrating Eq. (4) with the barred terms defined by Eqs. (10). The integration method used is identical to that used for the MDM. The displacements $u(t)$ are then calculated using Eq. (9).

Results and Discussion

Three example problems are presented to compare the convergence and computational time requirements of each of the four reduced-basis methods: the MDM, the MAM, the FDM, and the Lanczos method. The first example is a simply supported, multispan beam subject to a uniformly distributed load that varies as a quintic function of time. The second example is the simply supported, multispan beam subject to discrete damping. The third example is a high-speed civil transport subject to a normal wing-tip load that varies as a sinusoidal function of time. Results are presented that compare the number of basis vectors required by each of the methods to reach a predetermined error limit corresponding to a given value of the spatial error norm. For purposes of comparison, an error limit was established at a value of the error norm equal to 0.01, and the approximate solutions obtained using the reduced-basis methods are considered to be converged when the value of the error norm is equal to or less than this value. A discussion of the procedure used to establish this error limit is given in Ref. 12. Computational times for each of the methods are also presented and compared.

Undamped Multispan Beam Subject to a Uniformly Distributed Load

The multispan-beam example problem is illustrated in Fig. 1. The beam is simply supported and consists of 10 equally spaced spans. The finite element model of the beam contains 5 beam elements of equal length per span, for a total of 50 elements. Each of the corresponding 51 nodes has 2 degrees of freedom, and, excluding constraints, there is a total of 91 degrees of freedom. The dash pots in Fig. 1 are used to represent the location of the discrete translational and torsional dampers for the second example problem. The specifics of these dampers are given in the discussion of that example. The material properties, span length, and moment of inertia are all prescribed values of unity in English units: $E = 6895$ Pa (1 lb/in.^{2}), $\rho = 2.768 \times 10^4$ kg/m^3 (1 lbm/in.^{3}), $L = 0.0254$ m (1 in.), and $I = 4.162 \times 10^{-7}$ m^4 (1 in.^{4}). The forcing function $f(t)$ selected for this example varies as a quintic function of time and is defined by $f(t) = 1000(t^4 - t^5)$, where t is time. This forcing function was selected to allow the investigation of higher order forms of the FDM.[10] Since there are five nonzero time derivatives of the forcing function, any order method up to a sixth order may be used for the FDM.

This problem was selected as an example because it contains closely spaced natural frequencies. For this type of structure, the frequencies are closely spaced in groups equal in number to the number of spans in the beam (10 in this case). Furthermore, the mode shapes in each of these groups are similar, and the chances of a neglected higher mode having an effect on the response are greater. Hence, this problem enables the investigation of the ability of the higher order modal methods to approximate the effect of neglected higher modes.

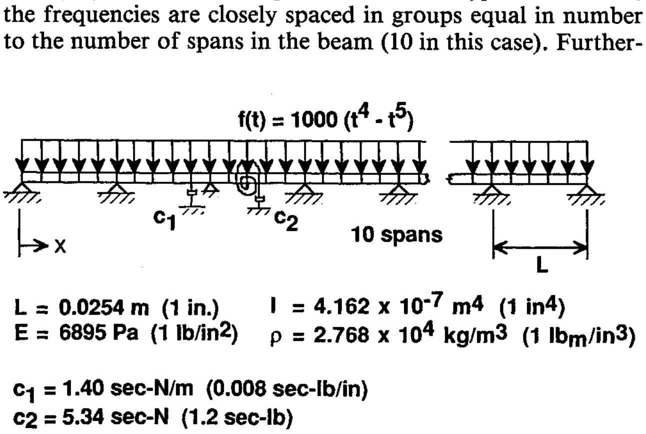

Fig. 1 Multispan beam subject to a uniformly distributed load that varies as a quintic function of time (not to scale).

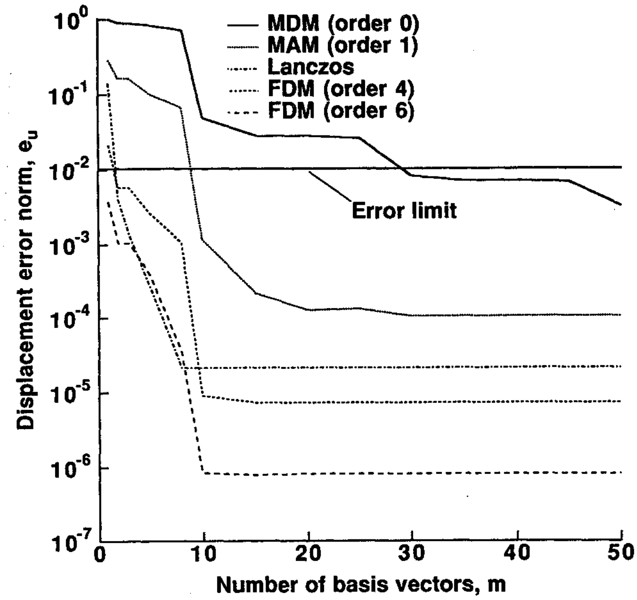

Fig. 2 Convergence of displacements for the undamped multispan beam illustrated in Fig. 1 (t = 1.2 s).

Convergence of Approximate Responses

A plot of the displacement error norm e_u as a function of the number of basis vectors m is shown in Fig. 2 for the multispan-beam example. The results shown in the figure are for the MDM, MAM, FDM (orders 4 and 6), and the Lanczos method at time $t = 1.2$ s. The solid horizontal line at a value of $e_u = 10^{-2}$ on the ordinate represents the value of the previously determined error limit.

As shown in Fig. 2, the FDM (order 6) converges using only one basis vector, and the FDM (order 4) and the Lanczos method each converge using two basis vectors. The MAM requires 10 basis vectors, and the MDM requires 30 basis vectors to converge to the prescribed error limit. The number of modes required for convergence of the MDM gives an indication of how many modes are important in the response of the beam. Furthermore, the results for the FDM illustrate how the higher order terms are able to approximate the effects of the higher modes that are neglected in the modal summation of Eq. (5). For this example problem, the Lanczos method is also able to adequately represent the displacement response with a small number of basis vectors.

The moment error norm e_M as a function of the number of basis vectors m is shown in Fig. 3. The results are presented at time $t = 1.2$ s for the same methods used in Fig. 2. Once again, the FDM (order 6) converges using only one basis vector, and the FDM (order 4) and the Lanczos method each require only two basis vectors. The MAM requires 10 basis vectors for convergence, and the MDM requires 61 basis vectors for convergence. The poor convergence of the moments exhibited by the MDM is not unusual since the moments depend on the second derivative of the displacements. However, these results demonstrate the ability of the higher order modal methods and the Lanczos method to accurately predict the moment response using a much reduced set of basis vectors.

Computational Time Requirements

A comparison of the computational time (i.e., CPU time) required for each method to yield converged displacement solutions is presented in Table 1. The CPU times presented are

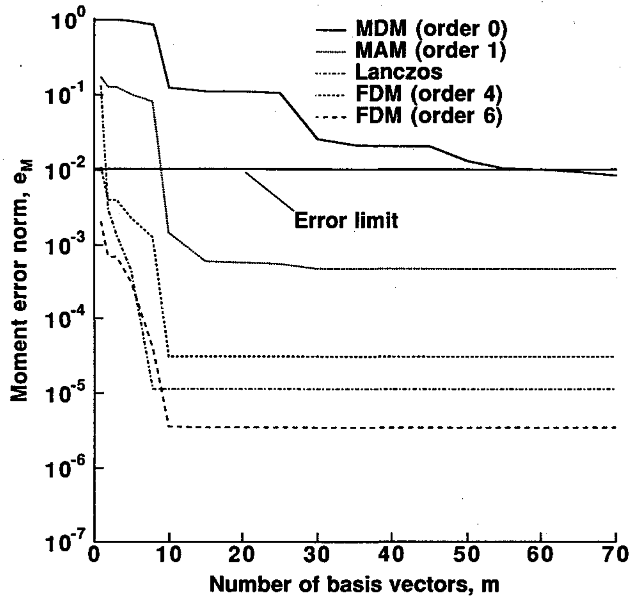

Fig. 3 Convergence of moments for the undamped multispan beam illustrated in Fig. 1 (t = 1.2 s).

for the first occurrence of a value of e_u less than or equal to the predefined error limit of 0.01. The number of basis vectors is the number required for each method to reach the value of e_u that is given in the table. The time shown for the Lanczos method includes the time for the eigensolver to compute the Lanczos vectors, and the times given for the modal methods include the time required for the computation of the indicated number of natural frequencies and eigenvectors using the Lanczos-vector-based eigensolver. As shown in Table 1, the time required for the FDM (orders 4 and 6) to converge is significantly smaller than that required for the full-system solution. The FDM (order 6) requires approximately one-half of the time required for the MAM and the Lanczos method, and it requires approximately one-third of the time required for the MDM. Furthermore, although the FDM (order 4) and the Lanczos method each require two basis vectors to converge, the FDM (order 4) requires less CPU time than the Lanczos method. Therefore, the CPU time advantage typically associated with the Lanczos method as compared with the MDM is not realized when using higher order forms of the FDM (orders 2 and higher). This time advantage associated with the Lanczos method is due to the relative lower cost of computing the Lanczos vectors as compared with computing the eigenvectors.[6] One reason that this time advantage is not realized is that the CPU time required for the Lanczos-vector-based eigensolver to compute the eigenvectors of the system is only slightly larger than that required to compute the Lanczos vectors, thus reducing the total CPU time required for the modal methods. A second reason that this time advantage is not realized is that the Lanczos method requires the solution of a tridiagonal system of equations, whereas the FDM requires the solution of an uncoupled system of equations. Additionally, although both the FDM (order 4) and the Lanczos method converge using two basis vectors, the FDM (order 6) provides a better approximation of the displacement response (i.e., a smaller value of e_u) using one basis vector.

A similar comparison of the CPU time required for each method to obtain converged moment solutions is presented in Table 2. The results are consistent with those shown in Table 1, with the FDM (order 6) requiring the smallest amount of CPU time. The FDM (order 6) also provides a better approximation of the moments than that provided by the FDM (order 4) and the Lanczos method with two basis vectors. These results indicate that the higher order modal methods do offer advantages over the other methods in terms of their requirements for basis vectors and computational time to achieve a converged solution.

Discretely Damped Multispan Beam Subject to a Uniformly Distributed Load

The discretely damped multispan-beam example problem is also illustrated in Fig. 1. The beam is identical to that used in the first example, with the exception of the added discrete dampers. As illustrated by the dash pots in Fig. 1, the beam is discretely damped with a translational damper of damping factor c_1 = 1.40 s-N/m (0.008 s-lb/in.) located at the node to the left of the third support, and a torsional damper of damping factor c_2 = 5.34 s-N (1.2 s-lb) located at the second node to the right of the third support. The material properties, span length, moment of inertia, and forcing function are all identical to those used for the first example. The reason this problem was studied was to obtain a comparison of these methods for a problem with the inclusion of damping. An added consideration is that since the damping is nonproportional, the equations of motion in second-order form [(Eq. (1)] cannot be uncoupled for the MDM, and the tridiagonal form of the Lanczos method is not preserved.

Convergence of Approximate Responses

A plot of the displacement error norm e_u as a function of the number of basis vectors m is shown in Fig. 4 for the discretely damped multispan beam example. The results shown in Fig. 4 are for the MDM, MAM, FDM (orders 4 and 6), and the Lanczos method at time t = 1.2 s. As shown in Fig. 4, the FDM (order 6) converges using only one basis vector, and the FDM (order 4) converges using two basis vectors. However, the MAM requires 42 basis vectors, the MDM requires 44 basis vectors, and the Lanczos method requires 49 basis vectors to converge to the prescribed error limit. It is apparent from the MDM results that a larger number of modes are important to the response of this discretely damped beam. Therefore, the results for the FDM (orders 4 and 6) illustrate once again that the higher order terms of the FDM are able to approximate the effects of the higher modes that are neglected in the modal summation of Eq. (5). However, the lower-order modal methods and the Lanczos method require a large number of basis vectors to adequately represent the displacement response.

The moment error norm e_M as a function of the number of basis vectors m is shown in Fig. 5. The results are presented at time t = 1.2 s for the same methods used in Fig. 4. The FDM (order 6) converges using only one basis vector, and the FDM (order 4) requires only two. However, the MDM, the MAM,

Table 1 CPU times at first occurrence of $e_u \leq 0.01$ for the undamped multispan beam example

Method	No. of basis vectors	$e_u \times 10^{-3}$	CPU time, s
MDM (order 0)	30	8.062	1.715
MAM (order 1)	10	1.126	1.094
FDM (order 4)	2	5.686	0.6246
FDM (order 6)	1	3.566	0.6040
Lanczos	2	4.037	1.112
Full-system solution	—	—	29.21

Table 2 CPU times at first occurrence of $e_M \leq 0.01$ for the undamped multispan beam example

Method	No. of basis vectors	$e_M \times 10^{-3}$	CPU time, s
MDM (order 0)	61	9.864	3.681
MAM (order 1)	10	1.396	1.314
FDM (order 4)	2	3.105	0.8446
FDM (order 6)	1	1.940	0.8240
Lanczos	2	2.848	1.332
Full-system solution	—	—	29.43

and the Lanczos method all require more than 70 basis vectors for a problem that has a total of only 91 degrees of freedom. These results demonstrate once again the ability of the higher order modal methods to accurately predict the moment response using a reduced set of basis vectors. The results also indicate that the addition of nonproportional damping can increase the number of basis vectors required by the Lanczos method for convergence. One reason for this increase in the number of basis vectors required for convergence may be that the discrete damping causes higher modes to participate in the response of the structure, thus requiring more Lanczos vectors to adequately approximate the response.

Computational Time Requirements

A comparison of the CPU time required for each method to yield converged displacement solutions is presented in Table 3. As with the first example, the time required for the FDM (orders 4 and 6) to converge is significantly smaller than that required for the full-system solution. The FDM (order 6) also requires approximately one-fourth of the time required for the MAM and the Lanczos method, and it requires slightly less than one-fifth of the time required for the MDM.

A similar comparison of the CPU time required for each method to obtain converged moment solutions is presented in Table 4. The results are consistent with those shown in Table 3, with the FDM (order 6) requiring the smallest amount of CPU time. Since the MDM, the MAM, and the Lanczos methods all require more than 70 basis vectors to converge, it may be concluded that for this example they offer minimal, if any, advantage over the direct integration of the complete system of equations of motion. However, the higher order FDM offers distinct advantages over the other methods in terms of its requirements for basis vectors and computational time.

Table 3 CPU times at first occurrence of $e_u \leq 0.01$ for the discretely damped multispan beam example

Method	No. of basis vectors	$e_u \times 10^{-3}$	CPU time, s
MDM (order 0)	44	9.740	10.07
MAM (order 1)	42	9.487	9.613
FDM (order 4)	2	6.094	2.386
FDM (order 6)	1	3.453	2.256
Lanczos	49	9.334	8.912
Full-system solution	—	—	45.51

Table 4 CPU times at first occurrence of $e_M \leq 0.01$ for the discretely damped multispan beam example

Method	No. of basis vectors	$e_M \times 10^{-3}$	CPU time, s
MDM (order 0)	70	28.31[a]	16.26
MAM (order 1)	70	27.04[a]	16.85
FDM (order 4)	2	4.091	2.606
FDM (order 6)	1	1.870	2.476
Lanczos	70	25.50[a]	14.54
Full-system solution	—	—	45.73

[a] e_M greater than prescribed limit of 0.01.

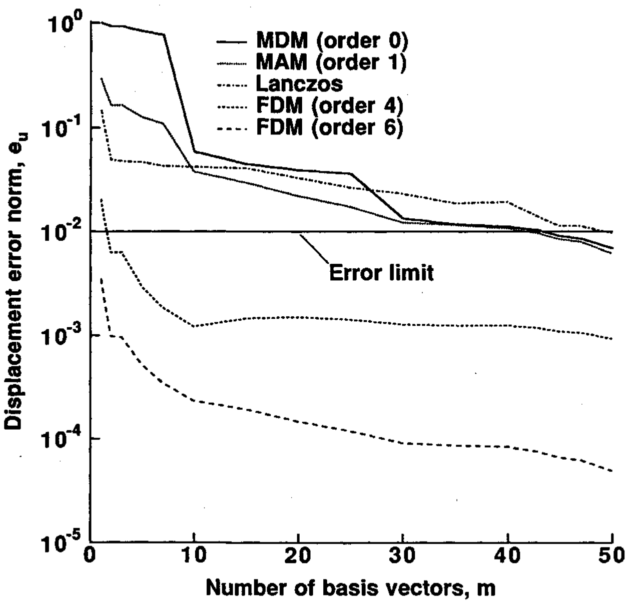

Fig. 4 Convergence of displacements for the discretely damped multispan beam illustrated in Fig. 1 (t = 1.2 s).

High-Speed Civil Transport Subjected to a Normal Wing-Tip Load

The high-speed civil transport problem is an example of a larger, more realistic structure and is illustrated in Fig. 6. Only one half of the structure is modeled to take advantage of the physical symmetry. A clamped boundary condition is imposed on the centerline running along the length of the model. The finite element model contains 398 nodes and a total of 1323 finite elements (rod, beam, membrane, and shear elements). Each of the nodes has three degrees of freedom, and, excluding constraints, there is a total of 972 degrees of freedom. The forcing function $f(t)$ acts normal to the wing tip (i.e., in the z direction), and the amplitude varies as a sinusoidal function of time defined by $f(t) = 1000 \sin(10t)$, where t is time. Damping is not considered in this example.

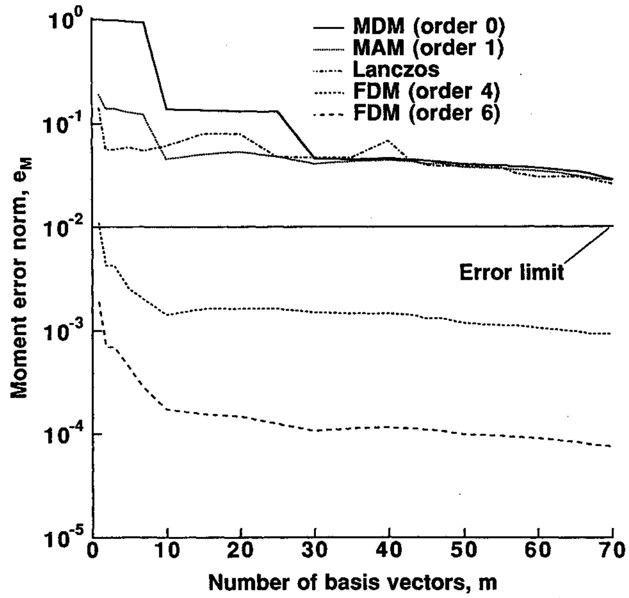

Fig. 5 Convergence of moments for the discretely damped multispan beam illustrated in Fig. 1 (t = 1.2 s).

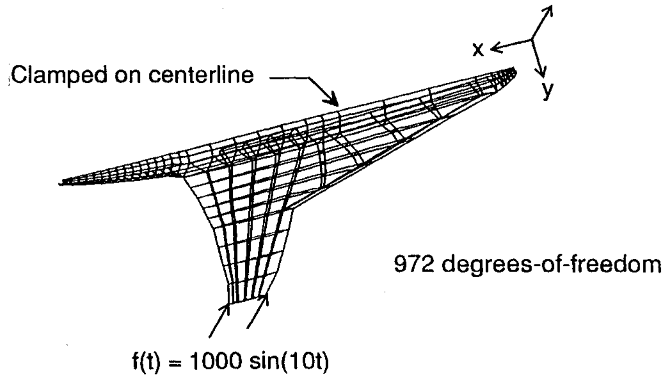

Fig. 6 High-speed civil transport subject to a normal wing-tip load that varies as a sinusoidal function of time.

Convergence of Approximate Responses

A plot of the displacement error norm e_u as a function of the number of basis vectors m is shown in Fig. 7 for the high-speed civil transport example. The results shown in the figure are for the MDM, MAM, FDM (orders 3 and 5), and the Lanczos method at time $t = 1.0$ s. The time $t = 1.0$ s was chosen because it is close to a maximum loading of the structure. As shown in Fig. 7, both the MAM and the FDM (orders 3 and 5) converge using only one basis vector. The Lanczos method requires three basis vectors, and the MDM requires four basis vectors for convergence. As indicated by the results for the MDM, only a small number of modes are important in the response of this structure.

A plot similar to Fig. 7 showing the stress error norm e_M as a function of the number of basis vectors m is given in Fig. 8. The values of stress used in Eq. (11) are the membrane stresses in the wing and fuselage cover panels. These cover panels are represented in the finite element model using membrane elements. The results are presented at time $t = 1.0$ s for the same methods used in Fig. 7. The MAM and the FDM (orders 3 and 5) each require 3 basis vectors to converge, the Lanczos method requires 8 basis vectors, and the MDM requires 35 basis vectors to converge.

Computational Time Requirements

A comparison of the CPU time required for each method to yield converged displacement solutions is presented in Table 5. For this example, the time required by all of the reduced-basis methods to achieve converged solutions is significantly less than that required for the full-system solution. However, since the MAM and FDM (orders 3 and 5) each require only one basis vector to converge, the MAM requires a slightly smaller amount of CPU time than the FDM (orders 3 and 5). The MAM also requires less than one-half of the time required by the Lanczos method and less than one-third of the time required by the MDM. Although both the MAM and the FDM (orders 3 and 5) require only one basis vector to converge, the FDM (order 5) provides a better approximation of the displacement stress response (i.e., lower error norm values) than does the MAM and FDM (order 3).

A comparison of the CPU time required for each method to obtain converged stress solutions is presented in Table 6. The results are consistent with those shown in Table 5, with the MAM requiring a slightly smaller amount of CPU time than the FDM (orders 3 and 5). In this case, since only the lower modes are important in the response, the CPU time advantage of the higher order modal methods is not as great as in the previous two examples. Therefore, the FDM (orders 3 and 5) gives better approximations than the MAM to the stress response for a given number of basis vectors, but it requires a slightly larger amount of CPU time than the MAM.

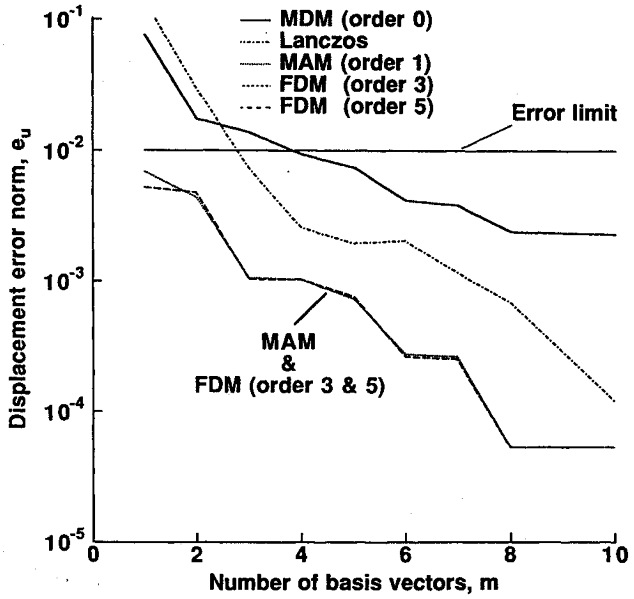

Fig. 7 Convergence of displacements for the high-speed civil transport illustrated in Fig. 6 ($t = 1.0$ s).

Table 5 CPU times at first occurrence of $e_u \leq 0.01$ for the high-speed civil transport example

Method	No. of basis vectors	$e_u \times 10^{-3}$	CPU time, s
MDM (order 0)	4	9.416	1.985
MAM (order 1)	1	6.855	1.678
FDM (order 3)	1	5.176	1.789
FDM (order 5)	1	5.125	1.904
Lanczos	3	7.243	5.433
Full-system solution	—	—	587.1

Table 6 CPU times at first occurrence of $e_s \leq 0.01$ for the high-speed civil transport example

Method	No. of basis vectors	$e_s \times 10^{-3}$	CPU time, s
MDM (order 0)	35	9.848	10.50
MAM (order 1)	3	8.574	3.057
FDM (order 3)	3	8.516	3.183
FDM (order 5)	3	8.516	3.325
Lanczos	8	5.517	7.231
Full-system solution	—	—	588.0

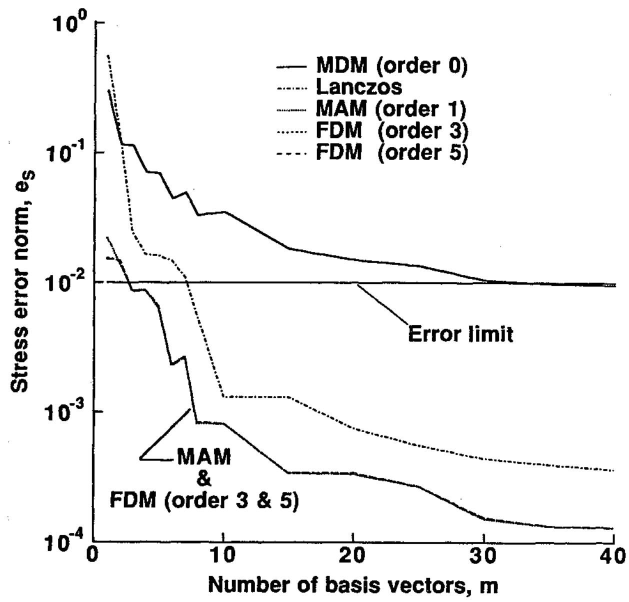

Fig. 8 Convergence of stresses for the high-speed civil transport illustrated in Fig. 6 ($t = 1.0$ s).

Concluding Remarks

The present study compares two advanced reduced-basis methods and two widely used lower order modal methods for linear, transient structural analysis. The advanced reduced-basis methods studied are a higher order modal method entitled the force-derivative method (FDM) and the Lanczos method, a reduced-basis method that uses Lanczos vectors as basis vectors. The lower order modal methods are the mode-displacement method (MDM) and the mode-acceleration method (MAM). Solutions from these four methods were compared with a full-system solution obtained by directly integrating the full system of equations of motion using the Newmark-Beta implicit time-integration scheme. The accuracy of the approximate responses was measured using a relative, spatial error norm. The number of basis vectors required to obtain a converged solution and the associated computational time for the four methods were compared. Three example problems were considered: an undamped and a discretely damped multispan beam subject to a uniformly distributed load that varies as a quintic function of time and a high-speed civil transport subject to a normal wing-tip load that varies as a sinusoidal function of time.

The FDM with the inclusion of both fourth- and sixth-order terms and the Lanczos method were shown to be very efficient methods for solving the undamped multispan beam example. This structure has closely spaced frequencies, and it was shown that a relatively high number of modes participate in the response of the beam. Therefore, the higher order forms of the FDM were effective in representing the effect of the higher modes that are neglected in the truncated form of the approximate solution. Although the Lanczos method obtained converged solutions using the same number of basis vectors as required by the FDM with fourth-order terms, the FDM with fourth-order terms required a smaller amount of CPU time for convergence. This CPU time savings occurred for two reasons: 1) the time required to calculate the eigenvectors of the system was only slightly larger than that required to compute the Lanczos vectors and 2) the FDM requires the solution of an uncoupled system of equations, whereas the Lanczos method requires the solution of a tridiagonal system of equations. The FDM with sixth-order terms produced the most accurate solutions with the fewest number of basis vectors, and it required the smallest amount of CPU time. The MAM was less efficient than these three methods in terms of the number of basis vectors and the amount of CPU time required to obtain converged solutions, and the MDM was the least efficient of all of the reduced-basis methods.

The FDM with the inclusion of both fourth- and sixth-order terms was also shown to be the most efficient method for solving the discretely damped multispan beam example. This structure also has closely spaced frequencies, and it was shown that the number of modes that participated in the response of this beam was higher than that for the undamped beam. Therefore, the FDM was very effective since it was once again able to represent the effects of the higher modes that are neglected in the approximate solution. The FDM was much more efficient than the MDM, the MAM, and the Lanczos method in terms of both the number of basis vectors required for convergence and computational time requirements. The FDM resulted in accurate solutions for displacements in approximately one-fourth the time required for the MAM and the Lanczos method and nearly one-fifth the time required for the MDM. The addition of discrete damping to this problem caused the Lanczos method to require a significantly higher number of basis vectors for convergence. The damping caused higher modes to participate in the response of the beam; therefore, more Lanczos vectors were required to adequately approximate the response of the beam.

For the high-speed civil transport example, all of the reduced-basis methods considered required a significantly smaller amount of CPU time to obtain converged displacement and stress solutions than that required for the full-system solution. Since only a small number of modes participated in the response of this structure, the MAM and the FDM with third- and fifth-order terms required the same number of basis vectors for convergence. These methods also required the fewest number of basis vectors for convergence; therefore, the MAM required the smallest amount of CPU time. The CPU time advantage of the higher order modal methods was not as great in this example as in the previous two examples because, although the FDM (orders 3 and 5) gave better approximations than the MAM to the displacement and stress responses for a given number of basis vectors, it required a slightly larger amount of CPU time than the MAM.

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